# Lecture 3

Spse Lie group $G$ acts transitively on set $X$.
Then $\forall x_0 \in X$, the map

$$\begin{aligned} G/H &\xrightarrow{\sigma_{x_0}} X \\ gH &\longmapsto g\cdot x_0 \end{aligned}$$

where $H = \mathrm{Stab}(x_0)$, is a bijection. So if $H$ is closed we get a smooth str on $X$ of dimension $\dim G - \dim H$. The "special" coset $eH$ corresp to $x_0 \in X$.

Every $x_0$ gives the same smooth structure, because the action of $G$ on $G/H$ is smooth.
(If $g\cdot x_0 = x_1$ then $\sigma_{x_1}^{-1} \circ \sigma_{x_0} : G/H \to G/H$ agrees w/ action of $g$.)

So we give $\mathrm{Flag}(V, \underline{d})$ this smooth str as

$\mathrm{Aut}(V)/P$ where $P = \{ g \in \mathrm{Aut}(V) \mid g\cdot E = E \}$

for some chosen base point $E \in \mathrm{Flag}(V, \underline{d})$.

Remaining tasks:

- Dimension formula
- Compactness.

Dimension. $\dim \mathrm{Aut}(V) = \beta n^2$ as $\mathrm{Aut}(V) \cong GL_n(k)$ is an open set in $k^{n^2}$.

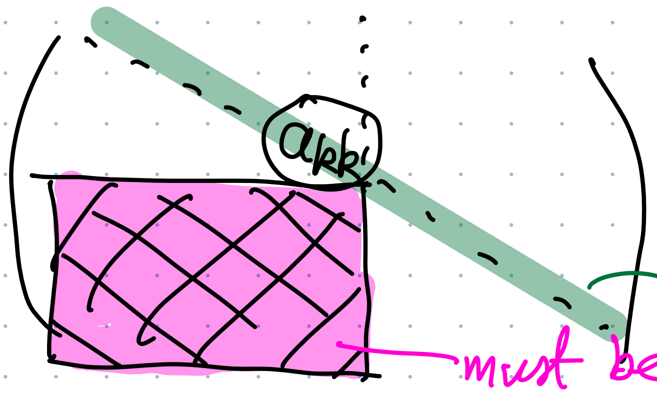
dim(P)? Need a descr of P.

**Full flag case:** $\underline{d} = (1, \dots, 1)$. $E \in Flag(V)$ base.

Let $e_1 \dots e_n$ be a flag basis for $E$.

**Lem.** $g \in Aut(V)$ has $g \cdot E = E$ iff its matrix $A$ relative to $e_1 \dots e_n$ has zeros below the main diag. (i.e. $A_{ij} = 0$ if $i < j$)

**Sketch** The requirement $g(\mathrm{span}(e_1 \dots e_k)) \subset \mathrm{span}(e_1 \dots e_k)$ is equivalent to zeros in this region:

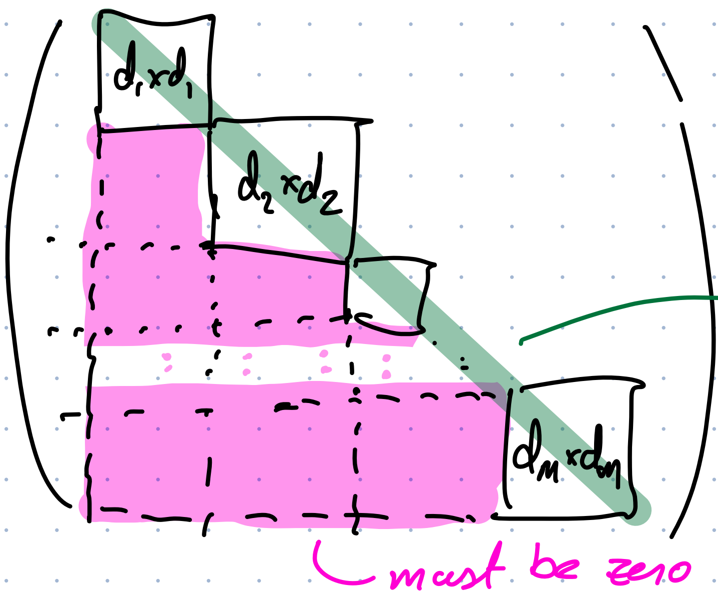



Letting k vary, union of rects is $\{i < j\}$

$\square$

The corresp statement for partial flags is

$P \cong \{$




$|$ invertible $\}$

**Note.** Usually sets of matrices w/ constrained entries are indicated by a template where "$*$" is used for an entry or block that can contain anything.

$$P \cong \left\{ \begin{pmatrix} \boxed{*} & & & * \\ & \boxed{*} & & \\ & & \ddots & \\ & & & \underbrace{\boxed{*}}_{d_m} \end{pmatrix} \right\}_{d_m}.$$

For dimension counting we can ignore the invertible assumption — but I prefer to state things in terms of tangent spaces.

$$\dim_{mfld} P = \dim_{k\text{-vec sp}} T_e P$$

$T_e P = \mathfrak{p}$ = the Lie algebra of $P \cong \{$left-inv vec fields on $P\}$

Then: $\mathrm{Lie}(\mathrm{Aut}(V)) = \mathrm{End}(V) \cong \begin{pmatrix} * \end{pmatrix}$ all $n \times n$ matrices over $k$.

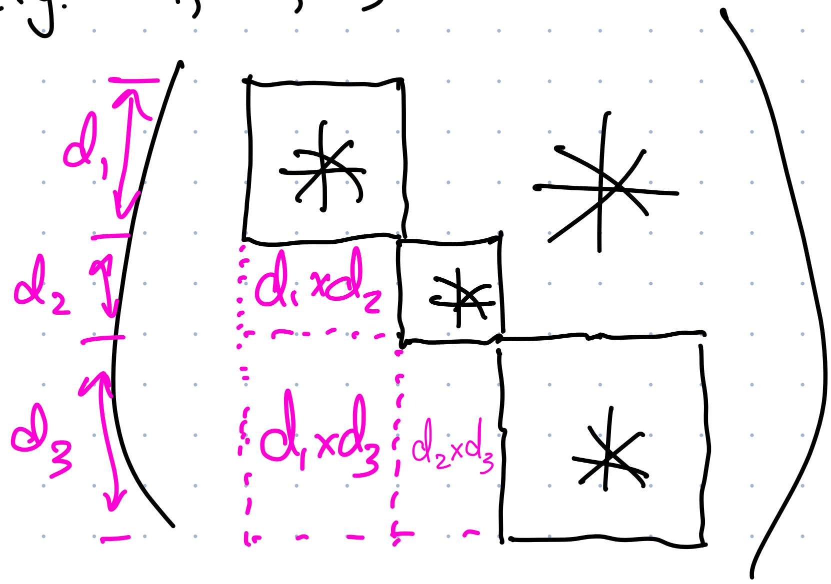
$$\mathfrak{p} = \mathrm{Lie}(P) \cong \begin{pmatrix} \boxed{*} & & & * \\ & \boxed{*} & & \\ & & \ddots & \\ & & & \boxed{*} \end{pmatrix}$$ all block-upper tri.

Now $\dim_k \mathrm{End}(V) - \dim_k(\mathfrak{p}) = \mathrm{codim}(\mathfrak{p})$

$= \#$ zero entries $= \displaystyle\sum_{1 \le i < j \le m} d_i d_j$

because the zero region is tiled by $d_i \times d_j$ rects.

e.g. $d_1, d_2, d_3$ case




**Compactness.** $k = \mathbb{R}$.

Introduce a positive definite inner product on $V$, $b$.
Then $O(V,b) = \{ g \in \mathrm{Aut}(V) \mid b(v,w) = b(g(v), g(w)) \ \forall v,w \}$ is a compact Lie group, because a $b$-orthonormal basis puts it in iso with

$O(n) = \{ A \in GL_n\mathbb{R} \mid A^T A = I \}$ a bdd closed set in $GL_n\mathbb{R}$

**Claim.** For any such inner product, $O(V,b)$ acts transitively on $\mathrm{Flag}(V, \underline{d})$.

<u>Pf.</u> Every flag has an orthonormal flag basis.
Any two orthonormal bases related by elt of $O(V,b)$. $\square$

**Cor.** $\mathrm{Flag}(V, \underline{d}) \cong O(V,b) / (P \cap O(V,b))$

And the image of $O(V,b)$ under the submersion

$$\begin{aligned} \mathrm{Aut}(V) &\twoheadrightarrow \mathrm{Flag}(V,\underline{d}) \\ g &\longmapsto g\cdot E \end{aligned}$$

is the entire space. Continuous image of a compact space is compact. ✓

$$\mathrm{Flag}(V,\underline{d}) \cong O(n) \big/ \left( O(d_1) \times O(d_2) \times \cdots \times O(d_m) \right)$$

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$k = \mathbb{C}$. Introduce herm innerproduct $h$.

$h(v,w) \in \mathbb{C}$ $\quad h(v,w) = \overline{h(w,v)}$ $\quad h(v,v) \geq 0$

$h(\lambda v, w) = \bar{\lambda}\, h(v,w)$. $\quad h(v_1+v_2, w) = h(v_1,w) + h(v_2,w)$

Preserved by $U(V,h)$ unitary grp

$\cong U(n) = \{ A \in GL_n\mathbb{C} \mid \bar{A}^T A = I \}$ compact.

Use $h$-orthonormal bases as above.

$$\mathrm{Flag}(V,\underline{d}) \cong U(n) \big/ U(d_1) \times \cdots \times U(d_n)$$

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Connectedness? $GL_n\mathbb{C}$ connected.

For $GL_n\mathbb{R}$, show $GL_n\mathbb{R}^+ = \{\det > 0\}$ acts transitively.

For $k = \mathbb{C}$, we even have $\pi_1(\mathrm{Flag}(V)) = 0$

Unlikely to get to this in lec #3

Pf. Fibration
$$\begin{array}{c} P \to G \\ \downarrow \\ G/P. \end{array}$$

$\rightsquigarrow$ $\pi_1(P) \to \pi_1(G) \to \pi_1(G/P) \to \pi_0(P)$

So it is enough to show

① $P$ is connected.

② $P \hookrightarrow G$ induces iso $\pi_1(P) \to \pi_1(G)$. (every loop in $G$ retracts to $U(n)$.

$$\begin{array}{l} U(1) \hookrightarrow U(n) \text{ iso on } \pi_1 \\ \cap \\ P \end{array}$$
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